

ASYMPTOTIC STABILITY OF PLANAR STATIONARY WAVES FOR TWO DIMENSIONAL SCALAR VISCOUS LAW

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Abstract

We investigate the asymptotic behaviours of the solutions to the initial-boundary value problem of scalar viscous conservation law in two space dimensions, with two boundaries. By virtue of an elementary energy method, both the global existence and asymptotic convergence toward the planar stationary waves are obtained for such an initial-boundary value problem.

1. Introduction and Main Results

Consider the initial-boundary value problem on a two-dimensional domain, denoted by $[0, 1] \times (-\infty, +\infty)$ for convenience, for the following scalar viscous conservation laws:

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$$\begin{cases} u_t + f(u)_x + g(u)_y = \Delta u, & (x, y) \in (0, 1) \times (-\infty, +\infty), \quad t > 0; \\ u(0, y, t) = u_-(t, y) \rightarrow u_-, \quad u(1, y, t) = u_+(t, y) \rightarrow u_+ (t \rightarrow +\infty), & t \geq 0; \\ u(x, y, 0) = u_0(x, y), & (x, y) \in [0, 1] \times [0, 1], \end{cases} \quad (1.1)$$

where u_{\pm} is given constant, which satisfy $u_- < u_+$, and $u_0(0, y) = u_-(0, y)$, $u_0(1, y) = u_+(1, y)$ for compatibility. We assume that $f, g \in C^2$ and

$$f''(u) > 0 \text{ for } u \text{ under consideration,} \quad (1.2)$$

and the boundary data $u_{\pm}(t, y)$ satisfies

$$\begin{cases} u_-(t, y) - u_-, u_+(t, y) - u_+ \in L^2; \\ \frac{\partial u_-(t, y)}{\partial t}, \frac{\partial u_+(t, y)}{\partial t} \in L^2 \cap L^1; \\ \frac{\partial^2 u_-(t, y)}{\partial t^2}, \frac{\partial^2 u_+(t, y)}{\partial t^2} \in L^1. \end{cases} \quad (1.3)$$

We recall that the stationary wave $\phi(x)$ of (1.1) is the unique solution of the following problem:

$$\begin{cases} f(\phi)_x = \phi_{xx}, & x \in (0, 1), \\ \phi(0) = u_-, & \phi(1) = u_+. \end{cases} \quad (1.4)$$

According to [17], we have the following Lemma 1 corresponding the existence of $\phi(x)$ to (1.4) for the non-degenerate case.

Lemma 1. *A necessary condition for the existence of solutions to the boundary value problem (1.4) is $f'(u_+) \leq 0$, we only consider the case $f'(u_+) < 0$. If $u_- < u_+$, there exists a monotone increasing solution $\phi(x)$, such that*

$$\left| \frac{\partial^k \phi(x)}{\partial x^k} \right| \leq C |u_- - u_+|, \quad (1.5)$$

for some constant C and $k \geq 0$ is an integer.

As previous works, the asymptotic behaviour of the corresponding Cauchy problem in one-dimensional space was discussed in [1-6], etc. To the initial-boundary value problem for scalar viscous conservation laws in one-dimensional space, the asymptotic behaviour of solutions has been considered by many authors (see [7-9], [13]).

This problem was also studied for the scalar viscous conservation laws in multi-dimensional whole space. For the related results, the interested reader is referred to [10, 14, 15, 16]. Whereas for the case of multi-dimensional conservation laws on half space, Kawashima et al. [11, 12] showed the asymptotic stability of stationary wave by using a weighted L^p energy method.

The main purpose of our present manuscript is devoted to the asymptotic stability of stationary wave to the initial-boundary value problem for the scalar viscous laws, on a two-dimensional domain. We do that by observing the following boundary function:

$$\begin{aligned} E(t, y) = & |u_-(t, y) - u_-| + (u_-(t, y) - u_-)^2 + |u_+(t, y) - u_+| \\ & + (u_+(t, y) - u_+)^2 + \left(\frac{\partial u_-(t, y)}{\partial t} \right)^2 + \left(\frac{\partial u_+(t, y)}{\partial t} \right)^2. \end{aligned} \quad (1.6)$$

Notations. Throughout this paper, we denote $[0, 1] \times (-\infty, +\infty)$ by K , K_x (K_y) denotes the projection of K in the x -direction (y -direction). $L^p = L^p(F)$ ($1 \leq p \leq \infty$) denotes the usual Lebesgue space on F with it's norm $\|f\|_{L^p} = \left(\int_F |f(x)|^p dx \right)^{1/p}$, $1 \leq p \leq \infty$, and when $p = 2$, we write $\|\cdot\|_{L^2} = \|\cdot\|$. H^1 denotes Sobolev space with it's norm $\|f\|_{H^1} = \|f\|_1 = \sqrt{\|f\|^2 + \|f_x\|^2 + \|f_y\|^2}$. We also use symbols

$$\begin{aligned}\nabla &= (\partial_x, \partial_y), \quad \nabla^2 = (\partial_x^2, \partial_x \partial_y, \partial_y \partial_x, \partial_y^2), \\ |\nabla u| &= \sqrt{u_x^2 + u_y^2}, \quad |\nabla^2 u| = \sqrt{u_{xx}^2 + u_{yy}^2 + u_{xy}^2 + u_{yx}^2},\end{aligned}$$

and for simplicity, $\|\nabla u\|$, $\|\nabla^2 u\|$ are denoted by $\|\nabla u\|$ and $\|\nabla^2 u\|$, respectively.

Our main results are as follows:

Theorem 1. *Suppose that (1.2) and (1.3) hold, and that $u_0(x, y) - \phi(x) \in H^1$, where ϕ is the stationary solutions obtained in Lemma 1. Then there exists a unique global solution u of (1.1) such that*

$$u - \phi \in C([0, \infty); H^1), \quad (u - \phi)_x, (u - \phi)_y \in L^2([0, \infty); H^1), \quad (1.7)$$

and

$$\sup_{(x, y) \in K} |u(x, y, t) - \phi(x)| \rightarrow 0, \text{ as } t \rightarrow +\infty. \quad (1.8)$$

To prove the convergence in Theorem 1, we need the following Lemma 2:

Lemma 2 ([18]). *Suppose that $g(t) \in L^1(0, +\infty)$, $g'(t) \in L^1(0, \infty)$, then*

$$\lim_{t \rightarrow \infty} g(t) = 0. \quad (1.9)$$

The rest of this paper is organized as follows: Section 2 is the proof of Theorem 1.

2. Convergence to Stationary Solutions

Now, we begin to prove Theorem 1. Let

$$u(x, y, t) = \phi(x) + v(x, y, t), \quad (2.1)$$

where ϕ is the stationary solution obtain in Lemma 1. Then we can reformulate (1.1) as follows:

$$\left\{ \begin{array}{l} v_t + (f(\phi + v) - f(\phi))_x + (g(\phi + v) - g(\phi))_y = \Delta v, \\ \quad (x, y) \in (0, 1) \times (-\infty, +\infty), \quad t > 0; \\ v(0, y, t) = u_-(t, y) - u_-, \quad v(1, y, t) = u_+(t, y) - u_+, \quad t \geq 0; \\ v(x, y, 0) = v_0(x) = u_0(x, y) - \phi(x), \quad (x, y) \in [0, 1] \times (-\infty, +\infty), \end{array} \right. \quad (2.2)$$

where $v(0, y, t), v(1, y, t) \rightarrow 0$, as $t \rightarrow +\infty$.

Theorem 3. *Assume that the same conditions as those in Theorem 1 hold (so $v_0 \in H^1$), then exists a unique solution of (2.2) satisfying*

$$v \in C([0, \infty); H^1), \quad v_x, v_y \in L^2([0, +\infty); H^1), \quad (2.3)$$

and

$$\sup_{(x, y) \in K} |v(x, y, t)| \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (2.4)$$

Theorem 1 is a direct consequence of Theorem 3, Theorem 3 can be proved by combining the local existence together with a priori estimates. For any $T > 0$, we seek the solution of (2.2) in the set of function $X_M(0, T)$ defined by

$$\begin{aligned} X_M(0, T) = \{ & v \in C([0, T]; H^1); \quad v_x, v_y \in L^2([0, T]; H^1); \\ & -\infty < \partial_x^m \partial_y^n v(x, y, t)|_{x=0,1, y=\pm\infty} < +\infty, \quad \forall t \in (0, T], m, n \in Z_+; \\ & \sup_{0 \leq t \leq T} \|v(t)\|_1 \leq M \}. \end{aligned}$$

We can easily show the local existence by a standard way, so we omit the details. What we have to do is to prove the following *a priori estimates*.

Proposition 1 (A priori estimate). *Suppose that v is a solution of (2.2) in $X_M(0, T)$ for a positive constant T . Then there exists a positive C independent of T , such that*

$$\begin{aligned} & \|v(t)\|_1^2 + \int_0^t (\|\sqrt{\phi_x}v(\tau)\|^2 + \|\nabla v(\tau)\|_1^2) d\tau \\ & \leq C(\|v_0\|_1^2 + \|E(t, y)\|_{L^1}), \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} E(t, y) &= |u_-(t, y) - u_-| + (u_-(t, y) - u_-)^2 + |u_+(t, y) - u_+| + (u_+(t, y) - u_+)^2 \\ &+ \left(\frac{\partial u_-(t, y)}{\partial t}\right)^2 + \left(\frac{\partial u_+(t, y)}{\partial t}\right)^2. \end{aligned}$$

Proof. Let v be a solution of (2.2) in $X_M(0, T)$. First, multiplying (2.2)₁ by v , we get

$$\begin{aligned} & \left(\frac{1}{2}v^2\right)_t + \{f(\phi + v)v - \int_\phi^{\phi+v} f(s)ds - v_x v\}_x + \{g(\phi + v)v - \int_\phi^{\phi+v} g(s)ds - v_y v\}_y \\ & + v_y^2 + (f(\phi + v) - f(\phi) - f'(\phi)v)\phi_x + v_x^2 = 0. \end{aligned} \quad (2.6)$$

By the maximum principle, $v_0 \in H^1$ and (1.3) give

$$\sup |v(x, y, t)| \leq C_0, \quad 0 \leq t \leq T, \quad (2.7)$$

where C_0 is a positive constant independent of T . Hence

$$(f(\phi + v) - f(\phi) - f'(\phi)v)\phi_x = \frac{1}{2}f''(y)v^2\phi_x \geq \frac{1}{2}D_0v^2\phi_x, \quad (2.8)$$

where y is between ϕ and $\phi + v$, $D_0 = \min_{u_- - C_0 \leq u \leq u_+ + C_0} f''(u) > 0$. And (2.2) gives

$$\int_R ((g(\phi + v))v - (\int_\phi^{\phi+v} g(s)ds - v_y v)_y) dy = 0. \quad (2.9)$$

Combine (2.8) and (2.9) together and integrate (2.6) over $[0, 1] \times (-\infty, +\infty) \times [0, t]$, we can change (2.6) into

$$\begin{aligned} & \|v(t)\|^2 + \int_0^t \{\|v_x(\tau)\|^2 + \|v_y(\tau)\|^2\} d\tau + C \int_0^t \{\|\sqrt{\phi_x} v(\tau)\|^2\} d\tau \\ & \leq I_1 + I_2 + C\|v(0)\|^2, \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} I_1 &= \int_0^t \int_R \{|v_x(0, y, \tau)v(0, y, \tau)| + |v_x(1, y, \tau)v(1, y, \tau)|\} dy d\tau, \\ I_2 &= \int_0^t \int_R \{|(u_+(\tau, y) - u_+)f(u_+)| + |\int_{u_+}^{u_+(\tau, y)} f(s) ds|\} dy d\tau \\ & \quad + \int_0^t \int_R \{|(u_-(\tau, y) - u_-)f(u_-)| + |\int_{u_-}^{u_-(\tau, y)} f(s) ds|\} dy d\tau. \end{aligned}$$

Since $f(\cdot)$ is bounded, I_2 can be estimated as

$$\begin{aligned} I_2 &\leq C\|u_+(t, y) - u_+\|_{L^1} + C\|u_-(t, y) - u_-\|_{L^1} \\ &\leq C\|E(t, y)\|_{L^1}. \end{aligned} \quad (2.11)$$

Next, we have I_1 estimated by Sobolev's inequality as follows:

$$\begin{aligned} I_1 &\leq \mu \int_0^t \int_R (v_x^2(0, y, \tau) + v_x^2(1, y, \tau)) dy d\tau \\ & \quad + C_\mu \int_0^t \int_R (v^2(0, y, \tau) + v^2(1, y, \tau)) dy d\tau \\ &\leq 2\mu \int_0^t \int_R \|v_x(x, y, \tau)\|_{L^2(K_x)} \|v_{xx}(x, y, \tau)\|_{L^2(K_x)} dy d\tau \\ & \quad + C_\mu \int_0^t \int_R [(u_+(\tau, y) - u_+)^2 + (u_-(\tau, y) - u_-)^2] dy d\tau \\ &\leq \mu \int_0^t \int_R \|v_x(x, y, \tau)\|_{L^2(K_x)}^2 dy d\tau + \mu \int_0^t \int_R \|v_{xx}(x, y, \tau)\|_{L^2(K_x)}^2 dy d\tau \\ &\leq \mu \int_0^t (\|\nabla^2 v(\tau)\|^2 + \|\nabla v(\tau)\|^2) d\tau + C\|E(t, y)\|_{L^1}, \end{aligned} \quad (2.12)$$

So, we obtain

$$\begin{aligned} & \|v(\tau)\|^2 + \int_0^t (\|\sqrt{\phi_x}v(\tau)\|^2 + \|\nabla v\|^2) d\tau \\ & \leq C(\|v_0\|^2 + \|E(t, y)\|_{L^1} + \mu_1 \int_0^t \|\nabla^2 v(\tau)\|^2 d\tau), \end{aligned} \quad (2.13)$$

here μ and μ_1 are sufficient small positive constant. Multiplying (2.6) by $-\Delta v$, gets

$$\begin{aligned} (\frac{1}{2}|\nabla v|^2)_t + |\nabla^2 v|^2 &= \Delta v((f(v+\phi) - f(\phi))_x + (g(v+\phi) - g(\phi))_y) \\ &+ (v_t v_y)_y + (v_t v_x)_x + 2(v_{xy} v_y)_x - 2(v_{xx} v_y)_y \\ &= F_0 + F_1 + F_2 + F_3 + F_4, \end{aligned} \quad (2.14)$$

where

$$F_0 = \Delta v((f(v+\phi) - f(\phi))_x + (g(v+\phi) - g(\phi))_y),$$

$$F_1 = (v_t v_y)_y, \quad F_2 = (v_t v_x)_x, \quad F_3 = 2(v_{xy} v_y)_x, \quad F_4 = -2(v_{xx} v_y)_y.$$

By Cauchy-Schwartz inequality and noticing that $f'(\cdot)$, $g'(\cdot)$, ϕ_x is bounded, one has

$$\begin{aligned} F_0 &= \Delta v(f'(v+\phi)\phi_x + f'(v+\phi)v_x - f'(\phi)\phi_x + g'(v+\phi)v_y) \\ &\leq \mu_2(\Delta v)^2 + C(f'^2(v+\phi)v_x^2 + g'^2(v+\phi)v_y^2) + C\phi_x v^2 \\ &\leq \mu_2(\Delta v)^2 + C|\nabla v|^2 + C\phi_x v^2 \\ &\leq \mu_3|\nabla^2 v|^2 + C|\nabla v|^2 + C\phi_x v^2. \end{aligned} \quad (2.15)$$

Next, we use (2.2) and Sobolev's inequality to estimate the F_i ($i = 1, 2, 3, 4$) as follows:

$$\int_0^t \int_R \int_0^1 F_1 dx dy d\tau = \int_0^t \int_0^1 [v_t v_y]_{y=-\infty}^{y=+\infty} dx d\tau = 0; \quad (2.16)$$

$$\int_0^t \int_R \int_0^1 F_3 dx dy d\tau = \int_0^t \int_R \int_0^1 F_4 dx dy d\tau = 0; \quad (2.17)$$

$$\begin{aligned} \int_0^t \int_R \int_0^1 F_2 dx dy d\tau &= \int_0^t \int_0^1 [v_t v_x]_{x=0}^{x=1} dy d\tau \\ &\leq \int_0^t \int_0^1 [Cv_t^2 + \mu_4 v_x^2]_{x=0}^{x=1} dy d\tau, \end{aligned}$$

similar to (2.12),

$$\begin{aligned} &\int_0^t \int_0^1 [Cv_t^2 + \mu_4 v_x^2]_{x=0}^{x=1} dy d\tau \\ &\leq C(\|\frac{\partial u_+(t, y)}{\partial t}\|_{L^2} + \|\frac{\partial u_-(t, y)}{\partial t}\|_{L^2}) + \mu_5(\int_0^t (\|\nabla^2 v(\tau)\|^2 + \|\nabla v(\tau)\|^2) d\tau) \\ &\leq C\|E(t, y)\|_{L^1} + \mu_5 \int_0^t \|\nabla^2 v(\tau)\|^2 d\tau + \mu_5 \int_0^t \|\nabla v(\tau)\|^2 d\tau. \end{aligned} \quad (2.19)$$

Here, $\mu_i (i = 2, 3, 4, 5)$ are sufficient small positive constants.

Combining (2.15)-(2.19) together and integrating (2.14) over $K \times [0, t]$, we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^t \int_R \int_0^1 (|\nabla v|^2)_t dx dy d\tau + \int_0^t \|\nabla^2 v(\tau)\|^2 d\tau \\ &\leq (\mu_3 + \mu_5) \int_0^t \|\nabla^2 v(\tau)\|^2 d\tau + C \int_0^t \|\nabla v(\tau)\|^2 d\tau + C\|E(t, y)\|_{L^1}, \end{aligned} \quad (2.20)$$

thus we have

$$\begin{aligned} &\|\nabla v(t)\|^2 + \int_0^t \|\nabla^2 v(\tau)\|^2 d\tau \\ &\leq C\|v_{0x}\|^2 + C\|v_{0y}\|^2 + C\|E(t, y)\|_{L^1} + C \int_0^t \|\nabla v(\tau)\|^2 d\tau. \end{aligned} \quad (2.21)$$

Combining (2.14) and (2.23), we can easily obtain

$$\begin{aligned}
& \|v(t)\|_1^2 + \int_0^t (\|\sqrt{\phi_x}v(\tau)\|^2 + \|\nabla v(\tau)\|_1^2) d\tau \\
& \leq C(\|v_0\|_1^2 + \|E(t, y)\|_{L^1}).
\end{aligned} \tag{2.22}$$

This completes the proof of the Proposition 1, so the global existence of the unique solution of (1.1) are obtained.

Next, we are going to show $\sup_{(x,y) \in K} |v(x, y, t)| = \sup_{(x,y) \in K} |u(x, y, t) - \phi(x)| \rightarrow 0$, as $t \rightarrow \infty$. We need the following estimates:

$$\begin{aligned}
v^2(x, y, t) &= 2 \int_0^x \frac{\partial}{\partial x} v^2(x_1, y, t) dx_1 + (u_-(t, y) - u_-)^2 \\
&\leq \int_0^1 v^2(x_1, y, t) dx_1 + \int_0^1 v_x^2(x_1, y, t) dx_1 + (u_-(t, y) - u_-)^2,
\end{aligned} \tag{2.23}$$

and we continue to estimates each term of the right side of (2.23),

$$\begin{aligned}
\int_0^1 v^2(x_1, y, t) dx_1 &= \int_{-\infty}^y \frac{\partial}{\partial y} \left(\int_0^1 v^2(x_1, y_1, t) dx_1 \right) dy_1 \\
&\leq 2 \int_{-\infty}^y \left(\int_0^1 |vv_y|(x_1, y_1, t) dx_1 \right) dy_1 \\
&\leq 2 \|v(t)\| \|v_y(t)\|,
\end{aligned} \tag{2.24}$$

similarly, we get

$$\int_0^1 v_x^2(x_1, y, t) dx_1 \leq 2 \|v_x(t)\| \|v_{xy}(t)\|. \tag{2.25}$$

So, we can now conclude that

$$\begin{aligned}
\sup v^2(x, y, t) &\leq 2 \|v(t)\| \|v_x(t)\| + 2 \|v_x(t)\| \|v_{xy}(t)\| \\
&\quad + (u_-(t, y) - u_-)^2.
\end{aligned} \tag{2.26}$$

Next, let $g(t) = \|v_x(t)\|^2$, then

$$\begin{aligned}
g'(t) &= \left(\int_{-\infty}^{+\infty} \int_0^1 v_x^2(x, y, t) dx dy \right)_t = 2 \int_{-\infty}^{+\infty} \int_0^1 v_x v_{xt}(x, y, t) dx dy \\
&= 2 \int_{-\infty}^{+\infty} [v_x v_t(x, y, t)]_{x=0}^{x=1} dy - 2 \int_{-\infty}^{+\infty} \int_0^1 v_{xx} v_t(x, y, t) dx dy \\
&\leq 2 \int_{-\infty}^{+\infty} ((u_-(t, y) - u_-)^2 + 2(u_+(t, y) - u_+)^2) dy \\
&\quad + 2 \int_{-\infty}^{+\infty} (v_x^2(1, y, t) + v_x^2(0, y, t)) dy - 2 \int_{-\infty}^{+\infty} \int_0^1 v_{xx} v_t(x, y, t) dx dy,
\end{aligned} \tag{2.27}$$

checking (2.12) again, we have

$$2 \int_{-\infty}^{+\infty} (v_x^2(1, y, t) + v_x^2(0, y, t)) dy \leq C(|\nabla^2 v(t)| + |\nabla v(t)|^2), \tag{2.28}$$

by (2.23)-(2.28),

$$2 \int_{-\infty}^{+\infty} (v_x^2(1, y, t) + v_x^2(0, y, t)) dy \in L^1_{(R_+)}, \tag{2.29}$$

by (1.3),

$$2(u_-(t, y) - u_-)^2 + 2(u_+(t, y) - u_+)^2 \in L^1, \tag{2.30}$$

by (2.2)₁ and (2.5),

$$\int_{-\infty}^{+\infty} \int_0^1 v_{xx} v_t(x, y, t) dx dy \in L^1_{(R_+)}. \tag{2.31}$$

So $g'(t) \in L^1_{(R_+)}$, thus, according to Lemma 2, $\lim_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow \infty}$

$\|v_x(t)\|^2 = 0$. Moreover, by (2.24) and (2.26) $\|v(t)\|, \|v_{xy}(t)\|$ is bounded about t , therefore by virtue of (2.28), we finally have

$$\lim_{t \rightarrow \infty} \sup_{(x, y) \in K} |v(x, y, t)| = 0. \tag{2.32}$$

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